Shot noise systems with random relaxations

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We explore a model of a random relaxation shot noise system in which (i) the shot inflow is a general Poisson point process, with possibly infinite Poissonian rates; and (ii) the exponential relaxations governing the shot decay are randomized. This system model is applicable to physical environments polluted by radioactive contamination of heterogeneous types. The statistics of random relaxation shot noise systems are analyzed quantitatively and comprehensively: stationary structure, correlation structure, process distribution, fractality and asymptotic fractality, and the display—both separately and simultaneously—of the Noah and Joseph effects. Results are obtained explicitly and in closed form, and facilitate the design of tractable shot noise systems with unique features.

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I. INTRODUCTION

Shot noise is the most fundamental quantitative model of discontinuous noise in continuous-time physical systems, and its research was pioneered by Campbell [1,2], Schottky [3], and Rice [4,5].

The classic model. In the classic shot noise model, external shots of random magnitude hit a physical system randomly in time. Shot arrivals follow a standard Poisson process, and shot magnitudes—upon impact—are independent and identically distributed (i.i.d.) positive-valued random variables drawn from a general, common, probability distribution. After impact, the shots undergo an exponential relaxation and decay to zero. The system's shot noise level $\xi(t)$ at time t equals the sum of all shot magnitudes present in the system at time t. The resulting shot noise process $[\xi(t)]_t$ is a *Markovian* process. Its dynamics are governed by the *Ornstein-Uhlenbeck* linear stochastic differential equation, driven by Poissonian noise.

Noah and Joseph effects. Named by Mandelbrot and Wallis, the Noah effect and the Joseph effect are used to describe fractal behavior of stationary random processes [6]. The former indicates "amplitudal fractality"—large amplitudal surges characterized by heavy-tailed (fat-tailed) stationary distributions [7]. The latter indicates "temporal fractality"—long-ranged temporal dependencies characterized by slowly decaying autocorrelations [8–10].

Classic shot noise processes can display the Noah effect, but cannot display the Joseph effect. Indeed, the exponential relaxation mechanism of classic shot noise systems always renders their autocorrelation functions exponential. In order to obtain shot noise processes displaying the Joseph effect, generalized relaxation mechanisms need to be considered. Two classes of shot noise system models capable of displaying both the Noah and Joseph effects, and generalizing the classic system model, are explored in (i) [11,12]—considering a linear non-Markovian relaxation mechanism; (ii) [13–15]—considering a nonlinear Markovian relaxation mechanism. (The intersection of these two classes of system models—namely, a linear Markovian relaxation mechanism—is the classic shot noise system model.)

Random relaxations. In this research we explore an alternative shot noise system model capable of displaying both the Noah and Joseph effects: *random relaxation shot noise*. In this system model we return to the classic exponential relaxation mechanism—adding one little twist: randomization. Specifically, each arriving shot, upon impact, picks at random its exponential relaxation mechanism. In other words, each arriving shot, upon impact, picks at random its half-life.

As will be demonstrated in the following, the incorporation of randomized relaxation has a dramatic effect on the resulting shot noise process $[\xi(t)]_t$: it changes the process's correlation structure profoundly, while having no effect whatsoever on the process's stationary structure. This amplitudal-temporal "orthogonality" is not achievable in the aforementioned generalized shot noise system models ([11,12] and [13–15]), and facilitates the design of shot noise systems with unique features.

Radioactive pollution. As a concrete example of random relaxation shot noise, consider a physical environment polluted by radioactive contamination. The contamination events are administered to the environment randomly in time, and are of varying sizes and heterogeneous types. The ith contamination event is characterized by three random "coordinates:" (i) the *time* at which the contamination took place; (ii) the *size* of the contamination; and (iii) the *half-life* of the contaminating material. The overall radioactivity level of the polluted environment is thus a shot noise process $[\xi(t)]_t$—albeit with randomly varying relaxation rates.

The difference between the cases of homogeneous pollution (where all contaminations are of the same type) and heterogeneous pollution (where the contaminations are of different types) is profound. Modeling the heterogeneous case via the classic shot noise model—rather than via the random relaxation shot noise model—leads to utterly incorrect results and predictions which grossly underestimate the long-term aftereffects of the radioactive pollution.

Inflow and analysis. In this research, we step beyond the conventional description and analysis of shot noise system models. In method, we follow [13–15] and consider the shot

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inflow to be a general Poisson point process (rather than a standard Poisson process). This allows the incorporation of shot inflows with *infinite* Poissonian rates, which, in turn, yields shot noise systems residing beyond the realm of classic shot noise systems—including, in particular, the class of fractal shot noise systems. In analysis, we follow [16–18] and undertake an approach that enables the simultaneous study of the Noah and Joseph effects.

A head-on simultaneous study of the Noah and Joseph effects is, by definition, impossible. Stationary processes exhibiting the Noah effect have infinite variances. Consequently, they fail to possess autocovariance functions—via which their temporal correlations are usually analyzed, and via which the Joseph effect is defined. Hence the study of the Joseph effect can be carried out only in the presence of finite variances—excluding all processes with heavy-tailed stationary distributions.

The methodology of *correlation cascades* was devised in [16,17] in order to quantitatively measure and characterize the temporal dependencies of random processes driven by infinite-variance Lévy noises, and was applied in [18] to the case of infinite-variance shot noise systems. This methodology further enabled the study of the ergodic properties of the random processes under consideration [19].

Following [18], a resolution-contingent Poissonian autocovariance function is introduced. This function (i) is easily computable and tractable; (ii) characterizes the *process distributions* of the random relaxation shot noise systems; and (iii) is always well defined—even in the case of divergent noise levels (let alone in the case of infinite-variance noise levels). The Poissonian autocovariance facilitates a precise quantitative analysis of random relaxation shot noise systems displaying, simultaneously, the Noah and Joseph effects.

Organization. The paper is organized as follows. We begin, in Sec. II, with the formulation of the random relaxation shot noise system model. Analysis of the shot noise stationary structure and correlation structure is conducted, respectively, in Secs. III and IV—followed by the analysis of the Noah and Joseph effects conducted in Sec. V. We conclude, in Sec. VI, with the study of *fractal* random relaxation shot noise systems.

A note about notation. Throughout the paper $\mathbf{I}\{S\}$ denotes the indicator function of the set S.

II. RANDOM RELAXATION SHOT NOISE

The random relaxation shot noise system model is formulated in this section: in Sec. II A, we introduce the model, and in Sec. II B, we describe the statistics of the underlying shot inflow. The notion of *Poisson point processes*, exploited in Sec. II B, is reviewed in the Appendix.

A. Model description

We consider a physical system which is stochastically perturbed by external shots. Shots are labeled by the index i, and each shot is described by three "impact coordinates:" t_i is the time epoch at which shot i hit the system (t_i real); x_i is the magnitude, upon impact, of shot i ($x_i>0$); y_i is the relaxation rate of shot i ($y_i>0$).

Each shot, right after impact, decays to zero—the dissipation of shot i being governed by the linear differential equation $\dot{X}=-y_iX$ (with initial level x_i). Hence, the magnitude of shot i, τ time units after impact, is $x_i\exp(-y_i\tau)$ ($\tau\geq 0$), and the half-life of shot i is $(\ln 2)/y_i$.

Consequently, the collection of all shot magnitudes present in the system at time t is given by

$$\Xi(t)=\{x_i\exp[-y_i(t-t_i)]\}_{t_i\leq t} \tag{1}$$

(t real). Equation (1), in turn, implies that the number of shot magnitudes present in the system at time t, residing above the resolution level l, is given by

$$N_l(t)=\sum_{t_i\leq t}\mathbf{I}\{x_i\exp[-y_i(t-t_i)]>l\} \tag{2}$$

(t real; $l>0$); and that the system's shot noise level at time t—i.e., the superposition of all shot magnitudes present in the system at time t—is given by

$$\xi(t)=\sum_{t_i\leq t}x_i\exp[-y_i(t-t_i)] \tag{3}$$

(t real).

Note that the integer $N_l(t)$ equals the *size* of the subcollection $\Xi(t)\cap(l,\infty)$, and that the shot noise level $\xi(t)$ equals the *sum* of the points of the collection $\Xi(t)$. Also note that the shot noise level $\xi(t)$ can be constructed from the resolution-contingent integers $\{N_l(t)\}_{l>0}$ via

$$\xi(t)=\int_0^\infty N_l(t)dl \tag{4}$$

(t real).

The *level processes* $N_l=[N_l(t)]_t$ ($l>0$) and the *shot noise process* $\xi=[\xi(t)]_t$ will be at the main focus of our analysis.

B. Shot inflow statistics

In the classic shot noise system model, the statistics of the underlying shot inflow are as follows.

(1) The shots' arrival epochs $\{t_i\}_i$ form a standard Poisson process with intensity η ($\eta>0$). In other words, the shots' interarrival periods form a sequence of i.i.d. positive-valued random variables drawn from a common exponential probability distribution with mean $1/\eta$.

(2) The shots' impact magnitudes $\{x_i\}_i$ form a sequence of i.i.d. positive-valued random variables drawn from a common probability distribution $\mathbf{F}_X$. (The arrival epochs $\{t_i\}_i$ and the impact magnitudes $\{x_i\}_i$ are independent.)

(3) The shots' relaxation rates $\{y_i\}_i$ all equal a common positive constant.

A natural generalization of the classic shot noise system model to the random relaxation shot noise system model is to consider the shots' relaxation rates $\{y_i\}_i$ as forming a sequence of i.i.d. positive-valued random variables drawn from a common probability distribution $\mathbf{F}_Y$. (The arrival epochs $\{t_i\}_i$, the impact magnitudes $\{x_i\}_i$, and the relaxation rates $\{y_i\}_i$ are independent.) This is equivalent to considering the collection of impact coordinates $\{(t_i,x_i,y_i)\}_i$ as forming a Poisson point process with Poissonian rate

$$\lambda(dt \times dx \times dy) = \eta\, dt\, \mathbf{F}_X(dx)\mathbf{F}_X(dy). \quad (5)$$

However, we go one step beyond Eq. (5) and consider the collection of impact coordinates $\{(t_i, x_i, y_i)\}_i$ as forming a Poisson point process with Poissonian rate

$$\lambda(dt \times dx \times dy) = dt\, \mu_X(dx)\mu_Y(dy), \quad (6)$$

where μ_X and μ_Y are *general measures* defined on the positive half line (rather than probability measures).

The difference between the shot inflow statistics of Eq. (5) and the shot inflow statistics of Eq. (6) is that the former allows only finite shot inflow rates whereas the latter also allows infinite shot inflow rates. The probability measures $\mathbf{F}_X$ and $\mathbf{F}_Y$ always assign a unit mass to the positive half line, and thus result in finite shot inflow rates. The general measures μ_X and μ_Y, on the other hand, may assign an infinite mass to the positive half line, and can thus result in infinite shot inflow rates. This difference, as we shall see in the following, has dramatic consequences and ramifications.

The measure μ_X governs the magnitudes of the incoming shots. Rather than dealing with the measure μ_X directly, we shall deal with its *tail function*:

$$\mathbf{T}(s) = \int_s^\infty \mu_X(dx) \quad (7)$$

$(s>0)$. The tail function $\mathbf{T}$ is monotonically decreasing to zero (as $s \to \infty$), and is bounded if and only if the measure μ_X equals (up to a multiplicative constant) a probability distribution $\mathbf{F}_X$. If the measure μ_Y is a probability measure (i.e., $\mu_Y = \mathbf{F}_Y$) then $\mathbf{T}(s)$ is the Poissonian rate at which shots with impact magnitude greater than size s hit the system.

The measure μ_Y governs the relaxation of the incoming shots. The measure μ_Y is henceforth assumed to satisfy the integrability condition $\int_0^\infty y^{-1}\mu_Y(dy) < \infty$. Rather than dealing with the measure μ_Y directly, we shall deal with its induced *distribution function*:

$$\mathbf{D}(s) = \int_0^s \frac{\kappa}{y}\mu_Y(dy) \quad (8)$$

$(s>0)$, where κ is a normalizing constant [given by $\kappa^{-1} = \int_0^\infty y^{-1}\mu_Y(dy)$]. The distribution function $\mathbf{D}$ is a probability distribution on the positive half line—increasing monotonically from zero (as $s \to 0$) to unity (as $s \to \infty$).

III. STATIONARY STRUCTURE

In this section we study the stationary statistical structure of the random relaxation shot noise system model. The analysis is based on the following proposition, whose proof is given in the Appendix.

Proposition 1. The collection of shot magnitudes $\Xi(t)$, present in the system at time t, is a Poisson point process on the positive half line with Poissonian rate

$$\lambda_\Xi(ds) = \left(\frac{1}{\kappa}\frac{\mathbf{T}(s)}{s}\right)ds. \quad (9)$$

Note that the Poissonian rate is independent of the time variable t—implying that the random relaxation shot noise system is *stationary*. With Proposition 1 at hand, the analysis of the level processes $\{N_l\}_{l>0}$ and of the shot noise process ξ follows straightforwardly.

A. The level processes

The definition of the level processes $\{N_l\}_{l>0}$, combined with Proposition 1, implies that the level processes are summable if and only if the function $\mathbf{T}(s)/s$ $(s>0)$ is integrable at infinity—in which case they are stationary, and the stationary distribution of the lth level process N_l is Poissonian with mean

$$\mathbf{M}(l) = \int_l^\infty \lambda_\Xi(ds) = \frac{1}{\kappa}\int_0^\infty \mathbf{T}(l \exp x)dx. \quad (10)$$

The function $\mathbf{M}(l)$ $(l>0)$, henceforth referred to as the *mean function*, will play a key role in the following.

B. The shot noise process

Campbell's theorem of the theory of Poisson processes (see [20], Sec. 3.2), combined together with Proposition 1, implies that the shot noise process ξ is summable if and only if the mean function $\mathbf{M}$ is integrable at the origin—in which case the shot noise process is stationary, and its stationary distribution is characterized by the Laplace transform

$$\mathbf{L}_\xi(\theta) = \exp\left(-\int_0^\infty [1-\exp(-\theta s)]\lambda_\Xi(ds)\right)$$
$$= \exp\left(-\theta\int_0^\infty \exp(-\theta l)\mathbf{M}(l)dl\right) \quad (\theta \geq 0). \quad (11)$$

The cumulants corresponding to the Laplace transform of Eq. (11)—which may happen to diverge—are given, in turn, by

$$\mathbf{C}_\xi(m) = \int_0^\infty s^m \lambda_\Xi(ds) = m\int_0^\infty l^{m-1}\mathbf{M}(l)dl \quad (12)$$

$(m=1,2,\ldots)$. The shot noise *mean* and *variance* equal, respectively, the first-order cumulant $\mathbf{C}_\xi(1)$ and the second-order cumulant $\mathbf{C}_\xi(2)$.

C. Random vs deterministic relaxation

The results obtained in this section—regarding the random relaxation shot noise stationary structure—turned out to be contingent on the relaxation mechanism μ_Y via the normalizing constant κ *alone*. That is, two random relaxation shot noise systems with the same shot inflow statistics (i.e., the same measures μ_X) but with different relaxation mechanisms (i.e., different measures μ_Y)—albeit with identical normalizing constant κ—will share the *same* stationary structure.

In particular, a classic shot noise system with relaxation rate κ and a random relaxation shot noise system with normalizing constant κ—both systems having the same shot inflow statistics—will display the same stationary structure. Thus, we conclude that *it is impossible to distinguish* *between classic shot noise systems and random relaxation shot noise systems via their stationary statistics*.

IV. CORRELATION STRUCTURE

In this section, we study the temporal statistical structure of the random relaxation shot noise system model. We begin with a proposition, whose proof is given in the Appendix, regarding the autocovariance functions of the level processes $\{N_l\}_{l>0}$ and the autocovariance function of the shot noise process ξ.

Proposition 2. (i) The autocovariance function of the lth level process N_l is given by

$$\mathbf{R}(\tau;l) = \int_0^\infty \mathbf{M}(l\exp(\tau y))\mathbf{D}(dy) \quad (\tau > 0). \tag{13}$$

(ii) The autocovariance function of the shot noise process ξ is given by

$$\mathbf{R}_\xi(\tau) = \mathbf{C}_\xi(2)\int_0^\infty \exp(-\tau y)\mathbf{D}(dy) \quad (\tau > 0). \tag{14}$$

Note that the level processes $\{N_l\}_{l>0}$, when summable, always posses well-defined autocovariance functions. This is because their stationary distributions—which are Poissonian—always posses finite variances. On the other hand, the shot noise process ξ, when summable, may or may not possess a well-defined autocovariance function—depending on whether or not its stationary distribution posses a finite variance [equaling its second-order cumulant $\mathbf{C}_\xi(2)$].

A. The Poissonian autocovariance

The autocovariance function $\mathbf{R}_\xi$ provides second-order information regarding the shot noise process ξ. The resolution-contingent autocovariance function $\mathbf{R}$—henceforth referred to as the *Poissonian autocovariance*—turns out to be far more informative.

Proposition 3. Let $-\infty < \tau_1 < \cdots < \tau_n < \infty$. The multidimensional probability generating function (PGF) of the random vector $(N_l(\tau_1), \ldots, N_l(\tau_n))$ is given by

$$\langle z_1^{N_l(\tau_1)} \cdots z_n^{N_l(\tau_n)}\rangle = \exp\left(\sum_{m=1}^{n}\sum_{k_1<\cdots<k_m} \mathbf{R}(\tau_{k_m} - \tau_{k_1};l)(z_{k_1}-1) \cdots (z_{k_m}-1)\right) \quad (|z_1|, \ldots, |z_n| \le 1). \tag{15}$$

Proposition 3, whose proof is given in the Appendix, implies that the multidimensional marginal distributions of the level processes $\{N_l\}_{l>0}$ are determined—via their corresponding multidimensional PGFs—by the Poissonian autocovariance $\mathbf{R}$. Hence, we conclude that *the Poissonian autocovariance* $\mathbf{R}$ *characterizes the process distribution of all level processes* $\{N_l\}_{l>0}$.

B. The Poissonian autocorrelation

The resolution-contingent autocorrelation function corresponding to the Poissonian autocovariance $\mathbf{R}$—and henceforth referred to, analogously, as the *Poissonian autocorrelation*—is given by

$$\rho(\tau;l) = \int_0^\infty \frac{\mathbf{M}(l\exp(\tau y))}{\mathbf{M}(l)}\mathbf{D}(dy) = \frac{\int_0^\infty \mathbf{T}(l\exp x)\mathbf{D}(x/\tau)dx}{\int_0^\infty \mathbf{T}(l\exp x)dx} \quad (\tau > 0). \tag{16}$$

The Poissonian autocorrelation ρ has three different interpretations: (i) it is the autocorrelation function of the level processes $\{N_l\}_{l>0}$; (ii) it is a Poissonian splitting ratio quantifying the shot noise temporal dependencies; and (iii) it is the survival probability of shot magnitudes above a given resolution level. For an explanation of the second and third interpretations, readers are referred to [18], Sec. V.

C. The shot noise autocorrelation

The autocorrelation function corresponding to the shot noise autocovariance function $\mathbf{R}_\xi$ is given by

$$\rho_\xi(\tau) = \int_0^\infty \exp(-\tau y)\mathbf{D}(dy) = \int_0^\infty \exp(-x)\mathbf{D}\left(\frac{x}{\tau}\right)dx \quad (\tau > 0). \tag{17}$$

Note that the function ρ_ξ may exist even in cases where the shot noise process ξ fails to be summable—let alone in cases where it fails to possess an autocorrelation function.

The function ρ_ξ turns out to be dependent, via the distribution function $\mathbf{D}$, on the relaxation mechanism μ_Y alone, and is independent of the tail function $\mathbf{T}$. Hence, the function ρ_ξ can be *reverse engineered* so as to yield desired correlation structures—the reverse engineering having no effect on the shot noise stationary structure. Examples of reverse engineered correlation structures are given in Table I.

V. NOAH AND JOSEPH EFFECTS

In this section, we study the Noah and Joseph effects as displayed by the random relaxation shot noise system under consideration (precise definitions of these effects will follow momentarily).

The analysis to be performed involves the notion of *regular variation* [21]. A real function ϕ is said to be regularly varying at the limit point p if the limit $\lim_{u\to p}\phi(\theta u)/\phi(u)$ exists for all positive constants θ. Theory shows that if the function ϕ is regularly varying then $\lim_{u\to p}\phi(\theta u)/\phi(u) = \theta^\nu$, where the exponent ν is a real parameter called the "exponent of regular variation." Regularly varying functions are generalizations of power laws, and play a key role in many fields of probability theory (see [21], Chap. 8).

TABLE I. Examples of reverse engineered correlation structures: The distribution function $\mathbf{D}$ required in order to yield a desired shot noise autocorrelation function ρ_ξ. The coefficient b and the exponent β are arbitrary positive parameters. In examples 2 and 3 the exponent β is restricted to the range $0<\beta<1$. The Mittag-Leffler function of example 3 is given by $E_\beta(z)=\Sigma_{k=0}^{\infty} z^k/\Gamma(1+\beta k)$; a probability distribution is Mittag-Leffler if its moment generating function is the Mittag-Leffler function. The compound Gamma distribution appearing in example 5 is obtained by the composition of two different and independent Gamma processes (in other words, by the subordination of one Gamma process to another, independent, Gamma process).

Example	Correlation structure	$\rho_\xi(\tau)=$	Distribution function $\mathbf{D}$
1	Exponential	$\exp(-b\tau)$	Deterministic
2	Stretched exponential	$\exp(-b\tau^\beta)$	Lévy stable
3	Mittag-Leffler	$E_\beta(-b\tau)$	Mittag-Leffler
4	Paretian	$(1+b\tau)^{-\beta}$	Gamma
5	Logarithmic	$[1+b\ln(1+\tau)]^{-\beta}$	Compound Gamma

A. The Noah effect

A positive-valued stationary random process is said to display the Noah effect [6] if the survival probability of its stationary distribution is regularly varying at infinity with exponent $\nu=-\alpha$, where $0<\alpha<1$. In other words, the process's stationary distribution is heavy tailed or fat tailed [7]—implying, in particular, that the stationary distribution is of infinite mean.

Let $\mathbf{S}_\xi$ denote the survival probability of the shot noise stationary distribution [whose Laplace transform is given in Eq. (11)]. We have the following proposition.

Proposition 4. Consider the tail function $\mathbf{T}$, the mean function $\mathbf{M}$, and the survival probability $\mathbf{S}_\xi$. If any of these functions is regularly varying at infinity with exponent $\nu=-\alpha$ ($\alpha>0$) then so are the others, and the following asymptotic equivalences hold:

$$\frac{1}{\alpha\kappa}\mathbf{T}(l) \underset{l\to\infty}{\sim} \mathbf{M}(l) \underset{l\to\infty}{\sim} \mathbf{S}_\xi(l). \tag{18}$$

In particular, Proposition 4 characterizes the case in which the shot noise process ξ displays the Noah effect ($0<\alpha<1$).

The proof of Proposition 4 is based on the theory of regular variation [applied to Eqs. (10) and (11)]: the implication from the tail function $\mathbf{T}$ to the mean function $\mathbf{M}$ is straightforward; the implication from the mean function $\mathbf{M}$ to the tail function $\mathbf{T}$ is due to Theorem 1.7.2 in [21]; the connection between the mean function $\mathbf{M}$ and the survival probability $\mathbf{S}_\xi$ is due to Theorem 8.2.1 in [21].

B. The Joseph effect

A finite-variance stationary random process is said to display the Joseph effect [6] if its autocorrelation function is regularly varying at infinity with exponent $\nu=-\beta$, where $0<\beta<1$. In other words, the process has long-range dependence or long-range memory [8–10]—implying, in particular, that the random process has an infinite range of dependence.[1]

For the random relaxation shot noise system we have the following proposition.

Proposition 5. The function ρ_ξ is regularly varying at infinity with exponent $\nu=-\beta$ if and only if the distribution function $\mathbf{D}$ is regularly varying at zero with exponent $\nu=\beta$ ($\beta>0$)—in which case the following asymptotic equivalences hold:

$$\rho_\xi(\tau) \underset{\tau\to\infty}{\sim} \Gamma(1+\beta)\mathbf{D}\left(\frac{1}{\tau}\right) \quad \text{and} \quad \mathbf{D}(s) \underset{s\to 0}{\sim} \frac{1}{\Gamma(1+\beta)}\rho_\xi\left(\frac{1}{s}\right). \tag{19}$$

Moreover, if the distribution function $\mathbf{D}$ is regularly varying at zero with exponent $\nu=\beta$ ($\beta>0$), then

$$\rho(\tau;l) \underset{\tau\to\infty}{\sim} \left(\frac{\int_0^\infty \mathbf{T}(l\exp x)x^\beta dx}{\int_0^\infty \mathbf{T}(l\exp x)dx}\right)\mathbf{D}\left(\frac{1}{\tau}\right). \tag{20}$$

In particular, Proposition 5 characterizes the case in which the shot noise process ξ—assuming that it is of finite variance—displays the Joseph effect ($0<\beta<1$).

The proof of Proposition 5 is based on the theory of regular variation [applied to Eqs. (16) and (17)]: the implication from the distribution function $\mathbf{D}$ to the function ρ_ξ and to the Poissonian autocorrelation ρ is straightforward; the implication from the function ρ_ξ to the distribution function $\mathbf{D}$ is due to Theorem 1.7.1′ in [21].

C. Noah and Joseph effects

The simultaneous study of the Noah and Joseph effects, however, is problematic. A stationary random process dis-

[1]The range of dependence of a finite-variance stationary random process is, by definition, the integral of its autocorrelation function. The range of dependence equals, up to a multiplicative constant, the value of the process's power spectrum at the origin.

playing the Noah effect is of infinite variance and thus fails to posses an autocorrelation function—via which the Joseph effect is defined. Hence, a simultaneous study of the Noah and Joseph effects in the case of the shot noise process ξ is impossible.

Nonetheless, the Poissonian autocorrelation ρ is well defined regardless of whether or not the shot noise process ξ displays the Noah effect. Thus, the Joseph effect can be addressed via the Poissonian autocorrelation ρ—specifically, via Eq. (20)—in cases where the Noah effect is displayed. We shall return to this point in Sec. VI.

D. An example

To illustrate the profound impact of random relaxations, consider the two following systems: (i) a classic shot noise system with deterministic relaxation rate κ; and (ii) a random relaxation shot noise system whose relaxation rates are drawn from a Gamma probability distribution with mode $m=\kappa$ and standard deviation $\sigma=(\sqrt{1+\beta}/\beta)\kappa$, where β is an arbitrary positive parameter.

Both these systems exhibit the very same stationary structure, namely, both systems share the same mean function $\mathbf{M}$. Yet the two systems have dramatically different correlation structures: in the first system the autocorrelation is exponential, $\rho_\xi(\tau)=\exp(-\kappa\tau)$, whereas in the second system the autocorrelation is algebraic,

$$\rho_\xi(\tau)=\frac{1}{[1+(\kappa/\beta)\tau]^\beta}\quad(\tau>0). \tag{21}$$

Note that the randomization of the relaxation rates in the second system was rather "mild." On the one hand, the distribution mode of the randomized relaxation rates was set to equal the deterministic relaxation rate of the classic shot noise system. On the other hand, the randomization chosen—the Gamma distribution—is a well-behaved distribution with finite moments of all orders and rapidly decaying probability tails.

One would expect that such mild randomization would have no more than mild effects on the resulting shot noise process ξ. Nonetheless, we see that this mild randomization generates long-term shot noise aftereffects. In environmental issues—such as the issue of heterogeneous radioactive pollution described in the introduction—long-term aftereffects are of considerable importance.

VI. FRACTAL SYSTEMS

Within the totality of all random relaxation shot noise systems there is a special class of fractal systems which we shall focus on in this section.

A. Fractality

Determining the class of fractal systems begins with the observation that the Poissonian autocorrelation ρ is independent of the resolution variable l if and only if the mean function $\mathbf{M}$ is a power law [this observation is an immediate consequence of Eq. (16)]. More specifically, we have the following proposition.

Proposition 6. The following statements are equivalent (a and α being arbitrary positive parameters).

(1) The tail function $\mathbf{T}$ admits the power-law form

$$\mathbf{T}(s)=\kappa\alpha\frac{a}{s^\alpha}\quad(s>0). \tag{22}$$

(2) The mean function $\mathbf{M}$ admits the power-law form

$$\mathbf{M}(l)=\frac{a}{l^\alpha}\quad(l>0). \tag{23}$$

(3) The Laplace transform $\mathbf{L}_\xi$ of the shot noise stationary distribution admits the Lévy stable form[2]

$$\mathbf{L}_\xi(\theta)=\exp[-\Gamma(1-\alpha)a\theta^\alpha]\quad(\theta\geq 0). \tag{24}$$

(4) The Poissonian autocorrelation ρ admits the resolution-independent form

$$\rho(\tau;l)=\rho_\xi(\alpha\tau)\quad(\tau\geq 0). \tag{25}$$

Proposition 6 follows straightforwardly from the results obtained in Secs. III and IV. We define a random relaxation shot noise system as *fractal* if any of the equivalent statements of Proposition 6 holds. The term "fractal" is due to the scale invariance of the Poissonian autocorrelation ρ with respect to the resolution variable l.

Note that fractal systems emanate from underlying measures μ_X assigning infinite mass to the positive half line. We stress that no classic shot noise system model—generating the initial shot magnitudes $\{x_i\}_i$ from a common probability distribution $\mathbf{F}_X$—can yield fractal shot noise systems.

B. Asymptotic fractality

Proposition 4 combined with Eq. (16) implies the following "asymptotic counterpart" of Proposition 6.

Proposition 7. The following statements are equivalent (α being an arbitrary positive parameter).

(1) The tail function $\mathbf{T}$ is regularly varying at infinity with exponent $\nu=-\alpha$.

(2) The mean function $\mathbf{M}$ is regularly varying at infinity with exponent $\nu=-\alpha$.

(3) The survival probability $\mathbf{S}_\xi$ is regularly varying at infinity with exponent $\nu=-\alpha$.

(4) The Poissonian autocorrelation ρ admits the limit:

$$\lim_{l\to\infty}\rho(\tau;l)=\rho_\xi(\alpha\tau)\quad(\tau\geq 0). \tag{26}$$

We define a random relaxation shot noise system as *asymptotically fractal* if any of the equivalent statements of Proposition 6 holds.

The limiting function $\rho(\tau;\infty)=\lim_{l\to\infty}\rho(\tau;l)$ $(\tau\geq 0)$ can be regarded as a "rare-event" autocorrelation function—measuring the shot noise correlations above high resolution levels $(l\to\infty)$.

[2]This statement holds only in the parameter range $0<\alpha<1$.

C. Noah and Joseph effects

Both fractal and asymptotically fractal systems with exponent $0<\alpha<1$ display the Noah effect. Yet in both such systems the function $\rho_\xi(\alpha\tau)$ $(\tau\geq 0)$ turns out to assume the role of a "legitimate" quantitative measure of correlation—though it cannot be interpreted as the autocorrelation of the shot noise process ξ (which is not defined due to infinite variances).

Thus, in the case of fractal and asymptotically fractal systems displaying the Noah effect, we can use Proposition 5 to characterize the display of the Joseph effect via the asymptotic behavior (in the limit $\tau\to\infty$) of the function $\rho_\xi(\alpha\tau)$.

D. Amplitudal-temporal decoupling

Fractal systems admit a unique amplitudal-temporal decoupling. This is best exemplified via the multidimensional PGFs of the level processes, for which

$$\langle z_1^{N_l(\tau_1)}\cdots z_n^{N_l(\tau_n)}\rangle = \exp[\mathbf{M}(l)\mathbf{P}_{\tau_1,\ldots,\tau_n}(z_1,\ldots,z_n)], \quad (27)$$

where

$$\mathbf{P}_{\tau_1,\ldots,\tau_n}(z_1,\ldots,z_n) = \sum_{m=1}^{n}\sum_{k_1<\cdots<k_m}\rho_\xi(\alpha(\tau_{k_m}-\tau_{k_1}))\times(z_{k_1}-1)\cdots(z_{k_m}-1)$$

$$(-\infty<\tau_1<\cdots<\tau_n<\infty;\ |z_1|,\ldots,|z_n|\leq 1). \quad (28)$$

The right-hand-side exponent of Eq. (27) factorizes into two terms: (i) The *amplitudal term* $\mathbf{M}(l)$, which depends on the resolution variable l alone, and which governs the system's stationary structure; (ii) the temporal term $\mathbf{P}_{\tau_1,\ldots,\tau_n}(z_1,\ldots,z_n)$, which depends on the time variables $\tau_1,\ldots,\tau_n$ alone, and which governs the system's temporal structure.

Thus, in the case of fractal systems, the amplitudal structure and the temporal structure decouple—the former being characterized by the mean function $\mathbf{M}$, and the latter being characterized by the function ρ_ξ. Such an amplitudal-temporal decoupling holds only in the case of fractal systems.

VII. CONCLUSIONS

We introduced and studied a random relaxation shot noise system model. This model generalizes the classic shot noise model in two ways: (i) it considers the shot inflow to follow a general Poissonian point process with possibly infinite Poissonian rates; and (ii) it randomizes the exponential relaxation mechanism governing the decay of incoming shots.

While retaining the linear and Markovian exponential relaxation of the classic shot noise model—a feature elemental in a plethora of physical systems—the random relaxation shot noise model is fully capable of displaying "anomalous statistics" which, in existing shot noise models, are attainable via the incorporation of either (i) linear non-Markovian relaxations, or (ii) nonlinear Markovian relaxations.

The results of this paper are summarized as follows.

(1) We analyzed both the stationary structure and the correlation structure of random relaxation shot noise systems, providing closed-form analytical formulas quantifying these structures.

(2) We facilitated the design of random relaxation shot noise systems whose stationary structure is identical to the stationary structure of classic shot noise systems—but yet their autocorrelation can be reverse engineered to admit any desired structure.

(3) We introduced a resolution-contingent Poissonian autocovariance function which turned out to characterize the process distribution of random relaxation shot noise systems.

(4) We showed that random relaxation shot noise systems can display both the Noah effect and the Joseph effect, and characterized the display—both separately and simultaneously—of these effects.

(5) We studied and characterized the class of (asymptotically) fractal random relaxation shot noise systems which are (asymptotically) invariant to changes in the resolution scale.

With regard to physical environments polluted by radioactive contamination, this research reveals a crucial difference between the cases of homogeneous pollution (where all contaminants are of the same type) and heterogeneous pollution (where the contaminants are of different types). While both cases exhibit identical radiation-level statistics, their long-term aftereffects are profoundly different: exponentially decaying correlations in the homogeneous case versus arbitrarily decaying correlations in the heterogeneous case. This understanding is of importance when assessing the impact of present-time pollutions on future radiation levels.

APPENDIX

In the appendix we prove the propositions asserted above. To that end we shall make use of the theory of Poisson point processes [20]. We define the notion of Poisson point processes, state two key results regarding these processes, and then turn to proving the propositions.

Definition. A countable collection of points $\{\omega_i\}_i$ scattered randomly across a space Ω is a Poisson point process with Poissonian rate $\lambda(d\omega)$ if (i) the number of points residing within the set $S\subset\Omega$ is a Poisson-distributed random variable with mean $\int_S\lambda(d\omega)$; and (ii) the numbers of points residing within disjoint sets are independent random variables.

Informally, this means that points are scattered randomly across the space Ω as follows. The space is divided into a countable collection of infinitesimal "space cells." At the cell $d\omega$—independently of all other cells—we toss a coin with infinitesimal success probability $\lambda(d\omega)$. If successful then a single point is placed at the cell, and if unsuccessful then the cell is left empty.

The characteristic functional. A countable collection of points $\{\omega_i\}_i$ scattered randomly across the space Ω is a Poisson point process with Poissonian rate $\lambda(d\omega)$ if and only if

$$\left\langle\prod_i f(\omega_i)\right\rangle = \exp\left(\int_\Omega[f(\omega)-1]\lambda(d\omega)\right) \quad (A1)$$

holds for all real test functions f [defined on the space Ω and such that the integral appearing on the right-hand side of

Eq. (A1) is convergent]. The left-hand side of Eq. (A1) is referred to as the *characteristic functional* of the point process $\{\omega_i\}_i$—the functional's variable being the test function f.

Covariance. Let $\{\omega_i\}_i$ be a Poisson point process on the space Ω with Poissonian rate $\lambda(d\omega)$, and let f and g be a pair of real functions defined on the space Ω. The covariance between the random sums $\Sigma_i f(\omega_i)$ and $\Sigma_i g(\omega_i)$ is given by

$$\mathrm{Cov}\left(\sum_i f(\omega_i), \sum_i g(\omega_i)\right) = \int_\Omega [f(\omega)g(\omega)]\lambda(d\omega), \quad \text{(A2)}$$

provided that the integral appearing on the right-hand side of Eq. (A2) is convergent.

1. Proof of Proposition 1

We compute the characteristic functional of the collection $\Xi(t)$.

Let f be an arbitrary test function defined on the positive half line. Then

$$\left\langle \prod_{p\in\Xi(t)} f(p) \right\rangle \quad \text{(A3)}$$

[due to the definition of the collection $\Xi(t)$—recall Eq. (1)]

$$=\left\langle \prod_{t_i\le t} f(x_i \exp[-y_i(t-t_i)]) \right\rangle \quad \text{(A4)}$$

[due to the fact that $\{(t_i,x_i,y_i)\}_i$ is a Poisson point process with Poissonian rate given by Eq. (6), and using Eq. (A1)]

$$=\exp\left(\int_{t'=-\infty}^{t}\int_{x=0}^{\infty}\int_{y=0}^{\infty}\{f(x\exp[-y(t-t')])-1\} \times dt'\mu_X(dx)\mu_Y(dy)\right). \quad \text{(A5)}$$

Now,

$$\int_{t'=-\infty}^{t}\int_{x=0}^{\infty}\int_{y=0}^{\infty}\{f(x\exp[-y(t-t')])-1\}dt'\mu_X(dx)\mu_Y(dy) \quad \text{(A6)}$$

(using the change of variables $\tau=t-t'$ and changing the order of integration)

$$=\int_{y=0}^{\infty}\left[\int_{x=0}^{\infty}\left(\int_{\tau=0}^{\infty}[f(x\exp(-y\tau))-1]d\tau\right)\mu_X(dx)\right]\mu_Y(dy) \quad \text{(A7)}$$

[using the change of variables $s=x\exp(-y\tau)$ in the inner integral]

$$=\int_{y=0}^{\infty}\left\{\int_{x=0}^{\infty}\left[\int_{s=0}^{\infty}\left(\frac{f(s)-1}{s}\right)\mathbf{I}\{s<x\}\frac{1}{y}ds\right]\mu_X(dx)\right\}\mu_Y(dy) \quad \text{(A8)}$$

(changing the order of integration)

$$=\left(\int_{y=0}^{\infty}\frac{1}{y}\mu_Y(dy)\right)\int_{s=0}^{\infty}[f(s)-1]\left(\frac{1}{s}\int_{x=s}^{\infty}\mu_X(dx)\right)ds \quad \text{(A9)}$$

[using the tail function $\mathbf{T}$ defined in Eq. (7) and the normalizing constant κ of Eq. (8)]

$$=\frac{1}{\kappa}\int_0^\infty [f(s)-1]\left(\frac{\mathbf{T}(s)}{s}\right)ds. \quad \text{(A10)}$$

Combining things together, we conclude that the characteristic functional of the collection $\Xi(t)$ is given by

$$\left\langle \prod_{p\in\Xi(t)} f(p) \right\rangle = \exp\left[\int_0^\infty [f(s)-1]\left(\frac{1}{\kappa}\frac{\mathbf{T}(s)}{s}\right)ds\right]. \quad \text{(A11)}$$

Equation (A1) thus implies that the collection $\Xi(t)$ is a Poisson point process on the positive half line with Poissonian rate

$$\lambda_\Xi(ds) = \left(\frac{1}{\kappa}\frac{\mathbf{T}(s)}{s}\right)ds. \quad \text{(A12)}$$

2. Proof of Proposition 2

We split the proof into two parts.

a. Level processes

We compute the autocovariance function $\mathbf{R}(\tau;l)$ $(\tau\ge 0)$ of the lth level process N_l:

$$\mathbf{R}(\tau;l) = \mathrm{Cov}[N_l(t),N_l(t+\tau)] \quad \text{(A13)}$$

[using Eqs. (2) and (A2)]

$$=\int_{t'=-\infty}^{\infty}\int_{x=0}^{\infty}\int_{y=0}^{\infty}(\mathbf{I}\{x\exp[-y(t-t')]>l\}\mathbf{I}\{t'\le t\} \times\mathbf{I}\{x\exp[-y((t+\tau)-t')]>l\} \times\mathbf{I}\{t'\le(t+\tau)\})dt'\mu_X(dx)\mu_Y(dy) \quad \text{(A14)}$$

(using the change of variables $u=t-t'$ and changing the order of integration)

$$=\int_{y=0}^{\infty}\left[\int_{u=0}^{\infty}\left(\int_{x=0}^{\infty}\mathbf{I}\{x>l\exp[y(u+\tau)]\}\mu_X(dx)\right)du\right]\mu_Y(dy) \quad \text{(A15)}$$

[using the tail function $\mathbf{T}$ defined in Eq. (7)]

$$= \int_{y=0}^{\infty} \left(\int_{u=0}^{\infty} \mathbf{T}(l \exp[y(u+\tau)]) du \right) \mu_Y(dy) \quad (A16)$$

(using the change of variables $s=l\exp[y(u+\tau)]$ in the inner integral)

$$= \int_{y=0}^{\infty} \left(\int_{s=l\exp(\tau y)}^{\infty} \frac{\mathbf{T}(s)}{s} ds \right) \frac{1}{y} \mu_Y(dy) \quad (A17)$$

[using the mean function $\mathbf{M}$ defined in Eq. (10) and the distribution function $\mathbf{D}$ defined in Eq. (8)]

$$= \int_{0}^{\infty} \mathbf{M}(l \exp(\tau y)) \mathbf{D}(dy). \quad (A18)$$

b. Shot noise process

We compute the autocovariance function $\mathbf{R}_\xi(\tau)$ $(\tau \geq 0)$ of the shot noise process ξ:

$$\mathbf{R}_\xi(\tau) = \mathrm{Cov}[\xi(t), \xi(t+\tau)] \quad (A19)$$

[using Eqs. (3) and (A2)]

$$= \int_{t'=-\infty}^{\infty} \int_{x=0}^{\infty} \int_{y=0}^{\infty} [x \exp[-y(t-t')] \mathbf{I}\{t' \leq t\} \times x \exp[-y((t+\tau)-t')] \times \mathbf{I}\{t' \leq (t+\tau)\}] dt' \mu_X(dx) \mu_Y(dy) \quad (A20)$$

(using the change of variables $u=t-t'$ and changing the order of integration)

$$= \left(\int_{x=0}^{\infty} x^2 \mu_X(dx) \right) \left[\int_{y=0}^{\infty} \left(\int_{u=0}^{\infty} \exp(-2yu) du \right) \times \exp(-\tau y) \mu_Y(dy) \right] \quad (A21)$$

[using, in the x integral, integration by parts and the tail function $\mathbf{T}$ defined in Eq. (7)]

$$= \left(2 \int_{s=0}^{\infty} s \mathbf{T}(s) ds \right) \left(\int_{y=0}^{\infty} \frac{1}{2y} \exp(-\tau y) \mu_Y(dy) \right) \quad (A22)$$

$$= \left[\int_{s=0}^{\infty} s^2 \left(\frac{1}{\kappa} \frac{\mathbf{T}(s)}{s} \right) ds \right] \left(\int_{y=0}^{\infty} \exp(-\tau y) \frac{\kappa}{y} \mu_Y(dy) \right) \quad (A23)$$

[using, in the s integral, integration by parts and the mean function $\mathbf{M}$ defined in Eq. (10); using, in the y integral, the distribution function $\mathbf{D}$ defined in Eq. (8)]

$$= \left(2 \int_{0}^{\infty} l \mathbf{M}(l) dl \right) \left(\int_{0}^{\infty} \exp(-\tau y) \mathbf{D}(dy) \right) \quad (A24)$$

[using Eq. (12) for the second-order cumulant $\mathbf{C}_\xi(2)$ of the shot noise stationary distribution]

$$= \mathbf{C}_\xi(2) \int_{0}^{\infty} \exp(-\tau y) \mathbf{D}(dy). \quad (A25)$$

3. Proof of Proposition 3

Let $-\infty < \tau_1 < \cdots < \tau_n < \infty$. We compute the multidimensional PGF of the random vector $(N_l(\tau_1), \ldots, N_l(\tau_n))$, and do so in three steps.

a. Step 1

Set

$$\chi_k(t,x,y) = \mathbf{I}\{x \exp[-y(\tau_k - t)] > l\} \mathbf{I}\{t \leq \tau_k\} \quad (A26)$$

$(k=1,\ldots,n;\ t$ real; $x,y>0)$. Combining Eqs. (2) and (A26) together, we have

$$N_l(\tau_k) = \sum_i \chi_k(t_i, x_i, y_i) \quad (A27)$$

$$\Rightarrow z_k^{N_l(\tau_k)} = \prod_i z_k^{\chi_k(t_i,x_i,y_i)} \quad (A28)$$

$$\Rightarrow \prod_{k=1}^{n} z_k^{N_l(\tau_k)} = \prod_i \left(\prod_{k=1}^{n} z_k^{\chi_k(t_i,x_i,y_i)} \right) \quad (A29)$$

$(|z_1|, \ldots, |z_n| \leq 1)$. Since $\{(t_i, x_i, y_i)\}_i$ is a Poisson point process with Poissonian rate given by Eq. (6), Eq. (A1) further implies that

$$\langle z_1^{N_l(\tau_1)} \cdots z_n^{N_l(\tau_n)} \rangle = \exp[\mathbf{P}_{\tau_1,\ldots,\tau_n}(z_1, \ldots, z_n)], \quad (A30)$$

where

$$\mathbf{P}_{\tau_1,\ldots,\tau_n}(z_1, \ldots, z_n) = \int_{t=-\infty}^{\infty} \int_{x=0}^{\infty} \int_{y=0}^{\infty} \left[\left(\prod_{k=1}^{n} z_k^{\chi_k(t,x,y)} \right) - 1 \right] \times dt\, \mu_X(dx) \mu_Y(dy). \quad (A31)$$

b. Step 2

Note that

$$\left(\prod_{k=1}^{n} z_k^{\chi_k(t,x,y)} \right) - 1 = \prod_{k=1}^{n} [1 + (z_k - 1)\chi_k(t,x,y)] - 1 = \sum_{m=1}^{n} \sum_{k_1 < \cdots < k_m} [\chi_{k_1}(t,x,y) \cdots \chi_{k_m}(t,x,y)] \times (z_{k_1} - 1) \cdots (z_{k_m} - 1), \quad (A32)$$

and hence

$$\mathbf{P}_{\tau_1,\ldots,\tau_n}(z_1, \ldots, z_n) = \sum_{m=1}^{n} \sum_{k_1 < \cdots < k_m} \mathcal{C}_{k_1,\ldots,k_m} \times (z_{k_1} - 1) \cdots (z_{k_m} - 1), \quad (A33)$$

where

$$\mathcal{C}_{k_1,\ldots,k_m}=\int_{t=-\infty}^{\infty}\int_{x=0}^{\infty}\int_{y=0}^{\infty}[\chi_{k_1}(t,x,y)\cdots\chi_{k_m}(t,x,y)]\times dt\mu_X(dx)\mu_Y(dy). \quad \text{(A34)}$$

Now, using Eq. (A26) and the fact that $\tau_{k_1}<\cdots<\tau_{k_m}$, we have

$$\begin{aligned}\chi_{k_1}(t,x,y)\cdots\chi_{k_m}(t,x,y) &= \mathbf{I}\{x\exp[-y(\tau_{k_1}-t)]>l\}\cdots\mathbf{I}\{x\exp[-y(\tau_{k_m}-t)]>l\}\\ &\quad\times\mathbf{I}\{t\le\tau_{k_1}\}\cdots\mathbf{I}\{t\le\tau_{k_m}\}\\ &= \mathbf{I}\{x\exp[-y(\tau_{k_m}-t)]>l\}\mathbf{I}\{t\le\tau_{k_1}\}\\ &= \mathbf{I}\{x>l\exp[y(\tau_{k_m}-t)]\}\mathbf{I}\{t\le\tau_{k_1}\}. \end{aligned} \quad \text{(A35)}$$

Equation (A35), in turn, implies that

$$\mathcal{C}_{k_1,\ldots,k_m}=\int_{t=-\infty}^{\infty}\int_{x=0}^{\infty}\int_{y=0}^{\infty}(\mathbf{I}\{x>l\exp[y(\tau_{k_m}-t)]\}\times\mathbf{I}\{t\le\tau_{k_1}\})dt\,\mu_X(dx)\mu_Y(dy). \quad \text{(A36)}$$

c. Step 3

We set $\Delta=\tau_{k_m}-\tau_{k_1}$ and compute the value of the coefficient $\mathcal{C}_{k_1,\ldots,k_m}$:

$$\int_{t=-\infty}^{\infty}\int_{x=0}^{\infty}\int_{y=0}^{\infty}(\mathbf{I}\{x>l\exp[y(\tau_{k_m}-t)]\}\mathbf{I}\{t\le\tau_{k_1}\})\times dt\,\mu_X(dx)\mu_Y(dy) \quad \text{(A37)}$$

(using the change of variables $\tau=\tau_{k_1}-t$ and changing the order of integration)

$$=\int_{y=0}^{\infty}\left[\int_{\tau=0}^{\infty}\left(\int_{x=0}^{\infty}\mathbf{I}\{x>l\exp[y(\Delta+\tau)]\}\mu_X(dx)\right)d\tau\right]\mu_Y(dy) \quad \text{(A38)}$$

[using the tail function $\mathbf{T}$ defined in Eq. (7)]

$$=\int_{y=0}^{\infty}\left(\int_{\tau=0}^{\infty}\mathbf{T}(l\exp[y(\Delta+\tau)])d\tau\right)\mu_Y(dy) \quad \text{(A39)}$$

(using the change of variables $s=l\exp[y(\Delta+\tau)]$)

$$=\int_{y=0}^{\infty}\left(\int_{s=l\exp(\Delta y)}^{\infty}\frac{\mathbf{T}(s)}{s}ds\right)\frac{1}{y}\mu_Y(dy) \quad \text{(A40)}$$

[using the mean function $\mathbf{M}$ defined in Eq. (10) and the distribution function $\mathbf{D}$ defined in Eq. (8)]

$$=\int_0^{\infty}\mathbf{M}(l\exp(\Delta y))\mathbf{D}(dy). \quad \text{(A41)}$$

Thus, using the definition of the Poissonian autocovariance [Eq. (13)], we obtain

$$\mathcal{C}_{k_1,\ldots,k_m}=\mathbf{R}(\tau_{k_m}-\tau_{k_1};l). \quad \text{(A42)}$$

Finally, combining Eqs. (A30), (A33), and (A42) together, we conclude that

$$\langle z_1^{N_l(\tau_1)}\cdots z_n^{N_l(\tau_n)}\rangle=\exp\left(\sum_{m=1}^{n}\sum_{k_1<\cdots<k_m}\mathbf{R}(\tau_{k_m}-\tau_{k_1};l)\cdot(z_{k_1}-1)\cdots(z_{k_m}-1)\right). \quad \text{(A43)}$$

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